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On the Characterization of Smarandache Curves of Fractional Order in Euclidean 3-Space

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Abstract

This paper investigates and characterizes Smarandache space curves, an important class of curves, using the Caputo fractional Frenet frame. The Frenet frame and fractional curvature functions have been calculated for these fractional Smarandache curves. To demonstrate the theoretical results obtained, an example of a fractional Smarandache curve derived from a helical curve is considered, and the curvatures of this curve are explicitly calculated. Finally, to show the effect of the fractional order parameter on the geometric behavior, a graphical analysis of the curvatures obtained for different fractional orders is presented.

Keywords: fractional derivative; curve; Smarandache curve; fractional Frenet frame

MSC: 53A04; 26A33

1. Introduction

The theory of curves constitutes one of the most fundamental structures in differential geometry. To understand the spatial motion of a solid object, it is characterized by a set of vectors embedded within the object, known as the Frenet apparatus [1,2]. For a curve locally lying in $\mathbb{R}^3$ that is differentiable at every point within its interval, these vectors can be constructed as a set of mutually perpendicular unit vectors. The local theory of space curves is encapsulated by the Serret–Frenet theorem, which describes the derivatives of the Frenet frame in terms of the frame itself. Moreover, the rates of change of these vectors along the curve give rise to the curvature functions [3]. Depending on the geometric character of a curve defined in three-dimensional Euclidean space, various alternative frames can be constructed. One significant alternative was proposed by Bishop in 1975 [4]. Known as the Bishop frame or parallel transport frame, it possesses advantageous properties compared to the standard Frenet frame [5,6]. The most notable of these is its behavior at points where the curve's derivative vanishes; at such singular points, the Frenet frame cannot be defined, whereas the Bishop frame remains well-defined and is instrumental in characterizing the curve's nature at that point [4–6]. Following Bishop's work, several other frames have been introduced, including the alternative moving frame [7], the Sasai frame [8], the Sabban frame [9], and the Flc frame [10]. Utilizing these frames, the characterizations of special classes of curves have been extensively reformulated [11–14].

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In recent years, a significant discussion has emerged regarding the possibility of constructing a theory of curves using fractional-order derivative operators instead of their integer-order counterparts [15–18]. Central to these discussions is the geometric interpretation of the fractional derivative operator, which has become a vibrant area of research [19–22]. Within this framework, numerous geometric approaches have been proposed; however, many proposed fractional derivative operators fail to satisfy the fundamental requirements of differential geometry [15–17,23]. Nevertheless, some operators do meet these criteria. Prominent examples include the conformal fractional derivative [19], the adapted Caputo fractional derivative [15], and the $\wedge$-fractional derivative operator introduced by Lazopoulos K. A. and Lazopoulos A. K. [17]. These operators possess essential properties such as linearity, Leibniz's rule, the chain rule, and the property that the derivative of a constant is zero, all of which are crucial for building a coherent theory of curves. Gözütok et al. [18] successfully reconstructed the Frenet frame and formulas using the conformal fractional derivative, and subsequently, many special curve classes have been characterized using this fractional Frenet framework [24,25]. Furthermore, by employing the fractional derivative operator introduced by Lazopoulos K. A. and Lazopoulos A. K., the Frenet frame and equations have been formulated in a fractional-order setting [17], leading to numerous publications on the subject [26,27]. More recently, the requirements of differential topology have been addressed using the adapted Caputo fractional derivative, enabling the characterization of curve theory in two- and three-dimensional Euclidean spaces from a fractional perspective [15,28–31].

In this paper, we revisit and characterize Smarandache space curves, an important class of curves, using the Caputo fractional Frenet frame developed by Yajima et al. We derive the Frenet frame and formulas for this specific type of fractional curve and compute its fractional curvature. Furthermore, we present an illustrative example of a fractional Smarandache curve derived from a helical curve and explicitly obtain its curvatures. Finally, we provide a graphical interpretation of these curvatures for various fractional orders.

2. Preliminaries

This section presents the fundamental definitions and theorems of fractional derivatives and integrals. Subsequently, the Caputo fractional Frenet frame and associated curvatures, as constructed by Yajima et al., are introduced [15].

Definition 1. *Let $f(s)$ be a continuous and differentiable function. The left and right Riemann–Liouville fractional integrals of order $\alpha > 0$ are defined as*

$$(I_{a^+}^{\alpha} f)(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s-\xi)^{\alpha-1} f(\xi)\, d\xi, \quad s > a,$$

$$(I_{b^-}^{\alpha} f)(s) = \frac{1}{\Gamma(\alpha)} \int_s^b (\xi-s)^{\alpha-1} f(\xi)\, d\xi, \quad s < b.$$

Here, $\Gamma(\alpha)$ denotes the Euler Gamma function, a special function given by [32]

$$\Gamma(\alpha) = \int_0^{\infty} e^{-s} s^{\alpha-1}\, ds.$$

Definition 2. *Let $f(s)$ be a continuous and differentiable function. The left-sided and right-sided Riemann–Liouville fractional derivatives are defined by*

$$(D_{a^+}^{\alpha} f)(s) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^s (s-\xi)^{-\alpha} f(\xi)\, d\xi, \quad s > a,$$

$$(D_{b^-}^{\alpha} f)(s) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_s^b (\xi - s)^{-\alpha} f(\xi)\, d\xi, \quad s < b,$$

where $0 < \alpha \leq 1$ *represents the order of the derivative [32].*

Definition 3. *Let* $f(s)$ *and* $g(s)$ *be differentiable functions on the finite interval* $[a, b]$*. The Caputo fractional derivative of order* α *of* $f(s)$ *is given by (see [32])*

$$({}^{C}D_{0^+}^{\alpha}) f(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^s (s - \xi)^{-\alpha} \frac{df(\xi)}{d\xi}\, d\xi.$$

Furthermore, the Caputo fractional derivative of the composite function $(f \circ g)(s)$ *and the Leibniz product rule are expressed, respectively, as [33]*

$$({}^{C}D_{0^+}^{\alpha})(f \circ g)(s) = \frac{f(g(s)) - f(g(0))}{\Gamma(1-\alpha)} s^{-\alpha} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{s^{n-\alpha}}{\Gamma(n-\alpha+1)} \frac{d^n f(g(s))}{ds^n},$$

$$({}^{C}D_{0^+}^{\alpha})(fg)(s) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{d^n f(s)}{ds^n} (D_{0^+}^{\alpha-n} g(s)) - \frac{s^{-\alpha}}{\Gamma(1-\alpha)} f(0) g(0).$$

As can be seen, the Caputo fractional derivative formula for the composite function given above is defined in the form of an infinite series period. However, this formulation makes it quite difficult to define geometric quantities such as curvature with the Frenet frame, which is one of the fundamental structures of the theory of curves. Yajima et al. construct the Frenet frame and the concept of curvature as follows, based on the case where $n = 1$, in order to obtain these fundamental geometric quantities within this infinite series formulation without compromising structural consistency [15]. Let us consider a regular space curve expressed as

$$\begin{aligned} \beta : I \to & \quad \mathbb{R}^3 \\ t \to & \quad \beta(t) = (x_1(t), x_2(t), x_3(t)) \end{aligned}$$

in the three-dimensional Euclidean space. For the case $n = 1$, the Caputo fractional composite derivative formulation is taken in the simplified form

$$\frac{d^{\alpha} \beta(t(s))}{ds^{\alpha}} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d\beta}{dt} \frac{dt}{ds} \tag{1}$$

and since the arc length parameter of the curve $\beta(t)$ is given by

$$u(t) = \int_0^t \sqrt{(\dot{x_1}(\xi))^2 + (\dot{x_2}(\xi))^2 + (\dot{x_3}(\xi))^2} d\xi, \tag{2}$$

the curve $\beta(t)$ transforms into the curve $\beta(s)$ as a fractional-order parametrization under the variable change

$$s = \left(\frac{\alpha^2}{\Gamma(2-\alpha)} u(t) \right)^{\frac{1}{\alpha}}. \tag{3}$$

From (2) and (3), it is found as

$$\frac{ds}{dt} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \left(\sqrt{(\dot{x_1}(t))^2 + (\dot{x_2}(t))^2 + (\dot{x_3}(t))^2} \right) \tag{4}$$

then, by applying the fractional derivative of order α to the curve $\beta(t)$, its written as

$$\frac{d^{\alpha} \beta(t(s))}{ds^{\alpha}} = \left(\frac{d^{\alpha} x_1(t)}{ds^{\alpha}}, \frac{d^{\alpha} x_2(t)}{ds^{\alpha}}, \frac{d^{\alpha} x_3(t)}{ds^{\alpha}} \right)$$

and if the norm is taken, it is obtained

$$\left\| \frac{d^{\alpha}\beta(t(s))}{ds^{\alpha}} \right\| = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{dt}{ds} \sqrt{\left(\frac{dx_1(t)}{dt}\right)^2 + \left(\frac{dx_2(t)}{dt}\right)^2 + \left(\frac{dx_3(t)}{dt}\right)^2} = 1 \tag{5}$$

according to Formulation (1) and (4). Thus, the tangent vector of the α order is defined as

$$\mathcal{E}_1^{\alpha}(s) = \left(\frac{d^{\alpha}x_1(s)}{ds^{\alpha}}, \frac{d^{\alpha}x_2(s)}{ds^{\alpha}}, \frac{d^{\alpha}x_3(s)}{ds^{\alpha}} \right).$$

Furthermore, since $\left\| \frac{d^{\alpha}\beta(t(s))}{ds^{\alpha}} \right\| = 1 \Rightarrow < \frac{d}{ds}(\mathcal{E}_1^{\alpha}(s)), \mathcal{E}_1^{\alpha}(s) >= 0$, a fractional-order principal normal unit vector of curve which is orthonormal to $\mathcal{E}_1^{\alpha}(s)$ is defined by $\mathcal{E}_2^{\alpha}(s) = \frac{\frac{d}{ds}(\mathcal{E}_1^{\alpha}(s))}{\left\| \frac{d}{ds}(\mathcal{E}_1^{\alpha}(s)) \right\|}$ and the binormal vector of fractional order is $\mathcal{E}_3^{\alpha}(s) = \mathcal{E}_1^{\alpha}(s) \wedge \mathcal{E}_2^{\alpha}(s)$. Thus, the vectors $\mathcal{E}_1^{\alpha}$, $\mathcal{E}_2^{\alpha}$ and $\mathcal{E}_3^{\alpha}$ form an orthonormal basis for three-dimensional Euclidean space. In addition, the fractional-order curvature and torsion functions of the β curve are obtained as

$$\kappa_{\alpha}(s) =< \frac{d\mathcal{E}_1^{\alpha}(s)}{ds}, \mathcal{E}_2^{\alpha}(s) > \quad \tau_{\alpha}(s) =< \frac{d\mathcal{E}_2^{\alpha}(s)}{ds}, \mathcal{E}_3^{\alpha}(s) >$$

respectively. For a more detailed discussion, see [15,16]. When a space curve is parameterized directly by α-arc length, it can be summarized by the following definition.

Definition 4 (see [15,16]). *The Caputo fractional Frenet frame, which constitutes the fundamental structure of fractional curve theory, is defined on a unit-speed fractional space curve parameterized by α-arc length in three-dimensional Euclidean space. The tangent vector $\mathcal{E}_1^{\alpha}(s)$, normal vector $\mathcal{E}_2^{\alpha}(s)$, and binormal vector $\mathcal{E}_3^{\alpha}(s)$ are given, respectively, by*

$$\mathcal{E}_1^{\alpha}(s) = \frac{d^{\alpha}\beta(s)}{ds^{\alpha}}, \quad \mathcal{E}_2^{\alpha}(s) = \frac{\frac{d}{ds}(\mathcal{E}_1^{\alpha}(s))}{\left\| \frac{d}{ds}(\mathcal{E}_1^{\alpha}(s)) \right\|}, \quad \mathcal{E}_3^{\alpha}(s) = \mathcal{E}_1^{\alpha}(s) \wedge \mathcal{E}_2^{\alpha}(s). \tag{6}$$

In addition, there exists a relationship between the fractional-order curvature and torsion functions of the curve β and standard integer-order curvature and torsion functions, given by

$$\kappa_{\alpha}(s) : I \to \mathbb{R}, \quad \kappa_{\alpha}(s) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} \kappa(s), \tag{7}$$

and

$$\tau_{\alpha}(s) : I \to \mathbb{R}, \quad \tau_{\alpha}(s) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} \tau(s). \tag{8}$$

Thus, the fractional Frenet formulas can be expressed in matrix form as

$$\frac{d}{ds} \begin{pmatrix} \mathcal{E}_1^{\alpha}(s) \\ \mathcal{E}_2^{\alpha}(s) \\ \mathcal{E}_3^{\alpha}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_{\alpha}(s) & 0 \\ -\kappa_{\alpha}(s) & 0 & \tau_{\alpha}(s) \\ 0 & -\tau_{\alpha}(s) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1^{\alpha}(s) \\ \mathcal{E}_2^{\alpha}(s) \\ \mathcal{E}_3^{\alpha}(s) \end{pmatrix}. \tag{9}$$

3. The Characterization of Fractional Smarandache Curves in Three-Dimensional Euclidean Space

In this section, we define various types of fractional Smarandache curves and compute their curvature and torsion functions in Euclidean 3-space. Additionally, an illustrative example is provided.

Definition 5. *Let β be a unit-speed fractional curve parameterized by α-arc length in three-dimensional Euclidean space, and let $\{\mathcal{E}_1^\alpha(s), \mathcal{E}_2^\alpha(s), \mathcal{E}_3^\alpha(s)\}$ be its fractional Frenet frame apparatus. The $\mathcal{E}_1^\alpha(s)\mathcal{E}_2^\alpha(s)$-fractional Smarandache curve is defined by*

$$\beta_{_I}(s) = \frac{\mathcal{E}_1^\alpha(s) + \mathcal{E}_2^\alpha(s)}{\sqrt{2}}, \tag{10}$$

where $0 < \alpha \leq 1$ and $s > 0$.

Theorem 1. *Let $\beta_{_I}(s) : I \to \mathbb{E}^3$ be a unit-speed fractional curve parameterized by α-arc length in three-dimensional Euclidean space. The Frenet frame of the first-type fractional Smarandache curve is given by*

$$\begin{aligned}
{}_{\mathbb{I}}\mathbb{E}_1^\alpha(s^*) &= \frac{-\kappa_\alpha(s)\mathcal{E}_1^\alpha(s) + \kappa_\alpha(s)\mathcal{E}_2^\alpha(s) + \tau_\alpha(s)\mathcal{E}_3^\alpha(s)}{\sqrt{2(\kappa_\alpha(s))^2 + (\tau_\alpha(s))^2}},\\
{}_{\mathbb{I}}\mathbb{E}_2^\alpha(s^*) &= \frac{\lambda_1\mathcal{E}_1^\alpha(s) + \lambda_2\mathcal{E}_2^\alpha(s) + \lambda_3\mathcal{E}_3^\alpha(s)}{\sqrt{(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2}},\\
{}_{\mathbb{I}}\mathbb{E}_3^\alpha(s^*) &= \frac{\begin{array}{r}(\kappa_\alpha(s)\lambda_3 - \tau_\alpha(s)\lambda_2)\mathcal{E}_1^\alpha(s)\\ + (\kappa_\alpha(s)\lambda_3 + \tau_\alpha(s)\lambda_1)\mathcal{E}_2^\alpha(s)\\ + (-\kappa_\alpha(s)\lambda_1 - \kappa_\alpha(s)\lambda_2)\mathcal{E}_3^\alpha(s)\end{array}}{\sqrt{\big(2(\kappa_\alpha(s))^2 + (\tau_\alpha(s))^2\big)\big((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\big)}},
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1 &= -\kappa(s)\kappa_\alpha(s)\big(2\kappa^2(s) + \tau^2(s)\big) + \tau(s)\big(\kappa(s)\tau'(s) - \tau(s)\kappa'(s)\big),\\
\lambda_2 &= -\kappa(s)\tau(s)\big(\tau'(s) + 2\kappa(s)\tau_\alpha(s) + \tau(s)\kappa_\alpha(s)\big)\\
&\quad + \tau^2(s)\big(\kappa'(s) - \tau(s)\tau_\alpha(s)\big) - 2\kappa^3(s)\kappa_\alpha(s),\\
\lambda_3 &= \kappa(s)\tau_\alpha(s)\big(2\kappa^2(s) + \tau^2(s)\big) + 2\kappa(s)\big(\kappa(s)\tau'(s) - \tau(s)\kappa'(s)\big),
\end{aligned} \tag{11}$$

and the prime denotes differentiation with respect to the integer-order parameter s. Furthermore, the curvature and torsion functions of the first-type fractional Smarandache curve are obtained as

$$\begin{aligned}
{}_{I}\kappa_\alpha^*(s) &= \frac{\big(\frac{s^*}{s}\big)^{\alpha-1}\sqrt{2}}{\big(2\kappa^2(s) + \tau^2(s)\big)^2}\sqrt{(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2},\\
{}_{I}\tau_\alpha^*(s) &= \frac{\big(\frac{s^*}{s}\big)^{\alpha-1}\sqrt{2}}{\big(2\kappa^2(s) + \tau^2(s)\big)\big((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\big)^2}\\
&\quad \times \Big(\big(\kappa(s)\lambda_3 - \tau(s)\lambda_2\big)A_1 + \big(\kappa(s)\lambda_3 + \tau(s)\lambda_1\big)A_2 + \big(-\kappa(s)\lambda_1 - \kappa(s)\lambda_2\big)A_3\Big)
\end{aligned}$$

with

$$\begin{aligned}
A_1 &= \Big(\big((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\big)\big(\lambda_1' - \lambda_2\kappa_\alpha(s)\big) - \lambda_1\big(\lambda_1\lambda_1' + \lambda_2\lambda_2' + \lambda_3\lambda_3'\big)\Big),\\
A_2 &= \Big(\big((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\big)\big(\lambda_1\kappa_\alpha(s) + \lambda_2' - \lambda_1\tau_\alpha(s)\big) - \lambda_2\big(\lambda_1\lambda_1' + \lambda_2\lambda_2' + \lambda_3\lambda_3'\big)\Big),\\
A_3 &= \Big(\big((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\big)\big(-\lambda_2\tau_\alpha(s) + \lambda_3'\big) - \lambda_3\big(\lambda_1\lambda_1' + \lambda_2\lambda_2' + \lambda_3\lambda_3'\big)\Big),
\end{aligned}$$

where s^* denotes the fractional arc length parameter given by

$$s^* = \left(\frac{\alpha^2}{\Gamma(2-\alpha)}\,u(s)\right)^{\frac{1}{\alpha}}, \tag{12}$$

and

$$u(s) = \int_0^s \left\| \frac{d\beta_I(\xi)}{d\xi} \right\| d\xi. \tag{13}$$

Proof. Let s^* be the α-fractional arc length parameterization of the first-type Smarandache curve defined above. Differentiating both sides of the curve equation with respect to s yields

$$\frac{d\beta_I(s^*)}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(\frac{d\mathcal{E}_1^\alpha(s)}{ds} + \frac{d\mathcal{E}_2^\alpha(s)}{ds}\right). \tag{14}$$

Using the relation

$$\frac{d\beta_I(s^*)}{ds^*} = \frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}\frac{d^\alpha \beta_I(s^*)}{d{s^*}^\alpha},$$

Equation (14) becomes

$$\frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}{}_{\mathbb{I}}\mathbb{E}_1^\alpha(s^*)\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\kappa_\alpha(s)\mathcal{E}_2^\alpha(s) - \kappa_\alpha(s)\mathcal{E}_1^\alpha(s) + \tau_\alpha(s)\mathcal{E}_3^\alpha(s)). \tag{15}$$

Taking the norm of both sides of Equation (15) gives

$$\frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}\frac{ds^*}{ds} = \sqrt{\frac{2(\kappa_\alpha(s))^2 + (\tau_\alpha(s))^2}{2}}.$$

Substituting this result into Equation (15) yields the fractional tangent vector

$${}_{\mathbb{I}}\mathbb{E}_1^\alpha(s^*) = \frac{-\kappa_\alpha(s)\mathcal{E}_1^\alpha(s) + \kappa_\alpha(s)\mathcal{E}_2^\alpha(s) + \tau_\alpha(s)\mathcal{E}_3^\alpha(s)}{\sqrt{2(\kappa_\alpha(s))^2 + (\tau_\alpha(s))^2}}. \tag{16}$$

Differentiating both sides of Equation (16) with respect to s leads to

$$\frac{d_{\mathbb{I}}\mathbb{E}_1^\alpha(s^*)}{ds^*}\frac{ds^*}{ds} = \frac{\lambda_1\mathcal{E}_1^\alpha(s) + \lambda_2\mathcal{E}_2^\alpha(s) + \lambda_3\mathcal{E}_3^\alpha(s)}{(2\kappa^2(s) + \tau^2(s))^{3/2}},$$

where $\lambda_1, \lambda_2, \lambda_3$ are as given in (11). Furthermore, using the expression

$$\frac{ds^*}{ds} = \frac{\sqrt{2\kappa^2(s) + \tau^2(s)}}{(s^*/s)^{\alpha-1}\sqrt{2}},$$

we obtain

$$\frac{d_{\mathbb{I}}\mathbb{E}_1^\alpha(s^*)}{ds^*} = \frac{(s^*/s)^{\alpha-1}\sqrt{2}}{\left(2\kappa^2(s) + \tau^2(s)\right)^2}\left(\lambda_1\mathcal{E}_1^\alpha(s) + \lambda_2\mathcal{E}_2^\alpha(s) + \lambda_3\mathcal{E}_3^\alpha(s)\right).$$

Consequently, the curvature function of the first-type fractional Smarandache curve is calculated as

$${}_I\kappa_\alpha^*(s) = \left\| \frac{d_{\mathbb{I}}\mathbb{E}_1^\alpha(s^*)}{ds^*} \right\| = \frac{(s^*/s)^{\alpha-1}\sqrt{2}}{\left(2\kappa^2(s) + \tau^2(s)\right)^2}\sqrt{(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2},$$

where s^* is given by Equation (12). The fractional normal vector is then found as

$${}_{\mathbb{I}}\mathbb{E}_2^\alpha(s^*) = \frac{d_{\mathbb{I}}\mathbb{E}_1^\alpha(s^*)/ds^*}{\left\|d_{\mathbb{I}}\mathbb{E}_1^\alpha(s^*)/ds^*\right\|} = \frac{\lambda_1\mathcal{E}_1^\alpha(s) + \lambda_2\mathcal{E}_2^\alpha(s) + \lambda_3\mathcal{E}_3^\alpha(s)}{\sqrt{(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2}}.$$

The fractional binormal vector is obtained via the cross product

$$\begin{aligned}{}_{\mathbb{I}}\mathbb{E}_3^{\alpha}(s^*) &= {}_{\mathbb{I}}\mathbb{E}_1^{\alpha}(s^*) \wedge {}_{\mathbb{I}}\mathbb{E}_2^{\alpha}(s^*) \\ &\quad (\kappa_\alpha(s)\lambda_3 - \tau_\alpha(s)\lambda_2)\mathcal{E}_1^{\alpha}(s) \\ &\quad + (\kappa_\alpha(s)\lambda_3 + \tau_\alpha(s)\lambda_1)\mathcal{E}_2^{\alpha}(s) \\ &\quad + (-\kappa_\alpha(s)\lambda_1 - \kappa_\alpha(s)\lambda_2)\mathcal{E}_3^{\alpha}(s) \\ &= \frac{}{\sqrt{\left(2(\kappa_\alpha(s))^2 + (\tau_\alpha(s))^2\right)\left((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\right)}}.\end{aligned}$$

Finally, since

$$\frac{d_{\mathbb{I}}\mathbb{E}_2^{\alpha}(s^*)}{ds^*} = \frac{(s^*/s)^{\alpha-1}\sqrt{2}\left(A_1\mathcal{E}_1^{\alpha}(s) + A_2\mathcal{E}_2^{\alpha}(s) + A_3\mathcal{E}_3^{\alpha}(s)\right)}{\left(\sqrt{2\kappa^2(s) + \tau^2(s)}\right)\left((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\right)^{\frac{3}{2}}}$$

the fractional torsion function is given by

$$\begin{aligned}{}_{I}\tau_{\alpha}^{*}(s) &= \left\langle \frac{d_{\mathbb{I}}\mathbb{E}_2^{\alpha}(s^*)}{ds^*}, {}_{\mathbb{I}}\mathbb{E}_3^{\alpha}(s^*) \right\rangle \\ &= \frac{(s^*/s)^{\alpha-1}\sqrt{2}}{\left(2\kappa^2(s) + \tau^2(s)\right)\left((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\right)^2} \\ &\times \left(\left(\kappa(s)\lambda_3 - \tau(s)\lambda_2\right)A_1 + \left(\kappa(s)\lambda_3 + \tau(s)\lambda_1\right)A_2 + \left(-\kappa(s)\lambda_1 - \kappa(s)\lambda_2\right)A_3 \right)\end{aligned}$$

where A_1, A_2, A_3 are as defined above. □

Definition 6. *Let β be a unit-speed fractional curve parameterized by α-arc length in three-dimensional Euclidean space, and let $\{\mathcal{E}_1^{\alpha}(s), \mathcal{E}_2^{\alpha}(s), \mathcal{E}_3^{\alpha}(s)\}$ be its fractional Frenet frame apparatus. The $\mathcal{E}_1^{\alpha}(s)\mathcal{E}_3^{\alpha}(s)$-fractional Smarandache curve is defined by*

$$\beta_{II}(s) = \frac{\mathcal{E}_1^{\alpha}(s) + \mathcal{E}_3^{\alpha}(s)}{\sqrt{2}}, \tag{17}$$

where $0 < \alpha \leq 1$ and $s > 0$.

Theorem 2. *Let $\beta_{II}(s) : I \to \mathbb{E}^3$ be a unit-speed fractional curve parameterized by α-arc length in three-dimensional Euclidean space. The Frenet frame of the second-type fractional Smarandache curve is given by*

$$\begin{aligned}{}_{\mathbb{II}}\mathbb{E}_1^{\alpha}(s^*) &= \pm\mathcal{E}_2^{\alpha}(s), \\ {}_{\mathbb{II}}\mathbb{E}_2^{\alpha}(s^*) &= \pm\frac{-\kappa(s)\mathcal{E}_1^{\alpha}(s) + \tau(s)\mathcal{E}_3^{\alpha}(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}, \\ {}_{\mathbb{II}}\mathbb{E}_3^{\alpha}(s^*) &= \frac{\tau(s)\mathcal{E}_1^{\alpha}(s) + \kappa(s)\mathcal{E}_3^{\alpha}(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}},\end{aligned}$$

and its curvature and torsion functions are obtained as

$$\begin{aligned}{}_{II}\kappa_{\alpha}^{*}(s) &= \frac{\sqrt{2}\sqrt{\kappa^2(s) + \tau^2(s)}\,\Gamma(2-\alpha)(s^*)^{\alpha-1}}{|\kappa(s) - \tau(s)|\,\alpha}, \\ {}_{II}\tau_{\alpha}^{*}(s) &= \frac{\sqrt{2}\left(\frac{s^*}{s}\right)^{\alpha-1}\left(\kappa(s)\tau'(s) - \tau(s)\kappa'(s)\right)}{|\kappa(s) - \tau(s)|\left(\kappa^2(s) + \tau^2(s)\right)},\end{aligned}$$

where s^* denotes the fractional arc length parameter given by

$$s^* = \left(\frac{\alpha^2}{\Gamma(2-\alpha)}\, u(s)\right)^{\frac{1}{\alpha}},$$

and

$$u(s) = \int_0^s \left\| \frac{d\beta_{II}(\xi)}{d\xi} \right\| d\xi. \tag{18}$$

Proof. Let s^* denote the α-fractional arc length parameterization of the second-type Smarandache curve defined above. Differentiating both sides of the curve equation with respect to s yields

$$\frac{d\beta_{II}(s^*)}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(\frac{d\mathcal{E}_1^\alpha(s)}{ds} + \frac{d\mathcal{E}_3^\alpha(s)}{ds}\right). \tag{19}$$

Using the relation

$$\frac{d\beta_{II}(s^*)}{ds^*} = \frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}\frac{d^\alpha \beta_{II}(s^*)}{d{s^*}^\alpha} = \frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}\, {}_{III}\mathbb{E}_1^\alpha(s^*),$$

Equation (19) becomes

$$\frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}\, {}_{III}\mathbb{E}_1^\alpha(s^*)\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\kappa_\alpha(s)\mathcal{E}_2^\alpha(s) - \tau_\alpha(s)\mathcal{E}_2^\alpha(s)). \tag{20}$$

Taking the norm of both sides of Equation (20) gives

$$\frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}\frac{ds^*}{ds} = \frac{|\kappa_\alpha(s) - \tau_\alpha(s)|}{\sqrt{2}}. \tag{21}$$

Substituting this result into Equation (20) yields the fractional tangent vector

$${}_{III}\mathbb{E}_1^\alpha(s^*) = \frac{(\kappa_\alpha(s) - \tau_\alpha(s))\mathcal{E}_2^\alpha(s)}{|\kappa_\alpha(s) - \tau_\alpha(s)|} = \pm\mathcal{E}_2^\alpha(s).$$

Differentiating the fractional tangent vector with respect to s and using Equation (21) we obtain

$$\frac{d\, {}_{III}\mathbb{E}_1^\alpha(s^*)}{ds^*} = \pm\frac{\sqrt{2}\,\Gamma(2-\alpha)(s^*)^{\alpha-1}}{|\kappa_\alpha(s) - \tau_\alpha(s)|\,\alpha}(-\kappa_\alpha(s)\mathcal{E}_1^\alpha(s) + \tau_\alpha(s)\mathcal{E}_3^\alpha(s)).$$

Consequently, the curvature function of the second-type fractional Smarandache curve is calculated as

$${}_{II}\kappa_\alpha^*(s) = \left\| \frac{d\, {}_{III}\mathbb{E}_1^\alpha(s^*)}{ds^*} \right\| = \frac{\sqrt{2}\sqrt{\kappa^2(s) + \tau^2(s)}\,\Gamma(2-\alpha)(s^*)^{\alpha-1}}{|\kappa(s) - \tau(s)|\,\alpha}.$$

The fractional normal vector is then found as

$${}_{III}\mathbb{E}_2^\alpha(s^*) = \frac{d\, {}_{III}\mathbb{E}_1^\alpha(s^*)/ds^*}{\|d\, {}_{III}\mathbb{E}_1^\alpha(s^*)/ds^*\|} = \pm\frac{-\kappa(s)\mathcal{E}_1^\alpha(s) + \tau(s)\mathcal{E}_3^\alpha(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}.$$

The fractional binormal vector is obtained via the cross product

$$\begin{aligned} {}_{III}\mathbb{E}_3^\alpha(s^*) &= {}_{III}\mathbb{E}_1^\alpha(s^*) \wedge {}_{III}\mathbb{E}_2^\alpha(s^*) \\ &= \frac{\tau(s)\mathcal{E}_1^\alpha(s) + \kappa(s)\mathcal{E}_3^\alpha(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}. \end{aligned}$$

Finally, since

$$\frac{d_{\text{III}}\mathbb{E}_2^{\alpha}(s^*)}{ds^*} = \frac{\sqrt{2}\left(\frac{s^*}{s}\right)^{\alpha-1}}{|\kappa(s)-\tau(s)|\left(\kappa^2(s)+\tau^2(s)\right)^{\frac{3}{2}}}\Big(\big(\kappa(s)\tau(s)\tau'(s)-\tau^2(s)\kappa'(s)\big)\mathcal{E}_1^{\alpha}(s)$$

$$+\big(-\tau(s)(\kappa(s)\kappa'(s)+\tau(s)\tau'(s))-(\kappa^2(s)+\tau^2(s))(\kappa(s)\kappa_{\alpha}(s)+\tau(s)\tau_{\alpha}(s))\big)\mathcal{E}_2^{\alpha}(s)$$

$$+\big(-\kappa(s)\kappa'(s)\tau(s)+\kappa^2(s)\tau'(s)\big)\mathcal{E}_3^{\alpha}(s)\Big)$$

the fractional torsion function is given by

$$\begin{aligned}{}_{II}\tau_{\alpha}^{*}(s) &= \left\langle \frac{d_{\text{III}}\mathbb{E}_2^{\alpha}(s^*)}{ds^*}, {}_{\text{III}}\mathbb{E}_3^{\alpha}(s^*)\right\rangle \\ &= \frac{\sqrt{2}\left(\frac{s^*}{s}\right)^{\alpha-1}\Big(\kappa(s)\tau'(s)-\tau(s)\kappa'(s)\Big)}{|\kappa(s)-\tau(s)|\left(\kappa^2(s)+\tau^2(s)\right)}.\end{aligned}$$

□

Definition 7. *Let β be a unit-speed fractional curve parameterized by α-arc length in three-dimensional Euclidean space, and let $\{\mathcal{E}_1^{\alpha}(s), \mathcal{E}_2^{\alpha}(s), \mathcal{E}_3^{\alpha}(s)\}$ be its fractional Frenet frame apparatus. The $\mathcal{E}_2^{\alpha}(s)\mathcal{E}_3^{\alpha}(s)$-fractional Smarandache curve is defined by*

$$\beta_{III}(s) = \frac{\mathcal{E}_2^{\alpha}(s)+\mathcal{E}_3^{\alpha}(s)}{\sqrt{2}}, \tag{22}$$

where $0 < \alpha \leq 1$ and $s > 0$.

Theorem 3. *Let $\beta(s) : I \to \mathbb{E}^3$ be a unit-speed fractional curve parameterized by α-arc length in three-dimensional Euclidean space. The Frenet frame of the third-type fractional Smarandache curve is given by*

$$\begin{aligned}{}_{\text{III}}\mathbb{E}_1^{\alpha}(s) &= \frac{-\kappa(s)\mathcal{E}_1^{\alpha}(s)-\tau(s)\mathcal{E}_2^{\alpha}(s)+\tau(s)\mathcal{E}_3^{\alpha}(s)}{\sqrt{\kappa^2(s)+2\tau^2(s)}},\\ {}_{\text{III}}\mathbb{E}_2^{\alpha}(s) &= \frac{\eta_1\mathcal{E}_1^{\alpha}(s)+\eta_2\mathcal{E}_2^{\alpha}(s)+\eta_3\mathcal{E}_3^{\alpha}(s)}{\sqrt{(\eta_1)^2+(\eta_2)^2+(\eta_3)^2}},\\ {}_{\text{III}}\mathbb{E}_3^{\alpha}(s) &= \frac{\begin{array}{c}\big(-\tau(s)\eta_3-\tau(s)\eta_2\big)\mathcal{E}_1^{\alpha}(s)\\ +\big(\kappa(s)\eta_3+\tau(s)\eta_1\big)\mathcal{E}_2^{\alpha}(s)\\ +\big(-\kappa(s)\eta_2+\tau(s)\eta_1\big)\mathcal{E}_3^{\alpha}(s)\end{array}}{\sqrt{\left(\kappa^2(s)+2\tau^2(s)\right)\left((\eta_1)^2+(\eta_2)^2+(\eta_3)^2\right)}},\end{aligned}$$

where

$$\begin{aligned}\eta_1 &= \kappa(s)\big(\kappa(s)\kappa'(s)+2\tau(s)\tau'(s)\big)+\big(\kappa^2(s)+2\tau^2(s)\big)\big(\tau(s)\kappa_{\alpha}(s)-\kappa'(s)\big),\\ \eta_2 &= \tau(s)\big(\kappa(s)\kappa'(s)+2\tau(s)\tau'(s)\big)+\big(\kappa^2(s)+2\tau^2(s)\big)\big(-\kappa(s)\kappa_{\alpha}(s)-\tau'(s)-\tau(s)\tau_{\alpha}(s)\big),\\ \eta_3 &= -\tau(s)\big(\kappa(s)\kappa'(s)+2\tau(s)\tau'(s)\big)+\big(\kappa^2(s)+2\tau^2(s)\big)\big(-\tau(s)\tau_{\alpha}(s)+\tau'(s)\big),\end{aligned}$$

and the prime denotes differentiation with respect to the integer-order parameter s. Furthermore, its curvature and torsion functions are obtained as

$$
\begin{aligned}
{}_{III}\kappa_{\alpha}^{*}(s) &= \frac{\left(\frac{s^{*}}{s}\right)^{\alpha-1}\sqrt{2}}{\left(\kappa^{2}(s)+2\tau^{2}(s)\right)^{2}}\sqrt{(\eta_1)^{2}+(\eta_2)^{2}+(\eta_3)^{2}},\\
{}_{III}\tau_{\alpha}^{*}(s) &= \frac{\left(\frac{s^{*}}{s}\right)^{\alpha-1}\sqrt{2}}{\left(\kappa^{2}(s)+2\tau^{2}(s)\right)\left((\eta_1)^{2}+(\eta_2)^{2}+(\eta_3)^{2}\right)^{2}}\left(B_1+B_2+B_3\right),
\end{aligned}
$$

with

$$
\begin{aligned}
B_1 &= \left(-\tau(s)\eta_3-\tau(s)\eta_2\right)\\
&\quad\times\left(\left((\eta_1)^{2}+(\eta_2)^{2}+(\eta_3)^{2}\right)\left(\eta_1'-\eta_2\kappa_{\alpha}(s)\right)-\eta_1\left(\eta_1\eta_1'+\eta_2\eta_2'+\eta_3\eta_3'\right)\right),\\
B_2 &= \left(\kappa(s)\eta_3+\tau(s)\eta_1\right)\\
&\quad\times\left(\left((\eta_1)^{2}+(\eta_2)^{2}+(\eta_3)^{2}\right)\left(\eta_1\kappa_{\alpha}(s)+\eta_2'-\eta_3\tau_{\alpha}(s)\right)-\eta_2\left(\eta_1\eta_1'+\eta_2\eta_2'+\eta_3\eta_3'\right)\right),\\
B_3 &= \left(-\kappa(s)\eta_2+\tau(s)\eta_1\right)\\
&\quad\times\left(\left((\eta_1)^{2}+(\eta_2)^{2}+(\eta_3)^{2}\right)\left(\eta_2\tau_{\alpha}(s)+\eta_3'\right)-\eta_3\left(\eta_1\eta_1'+\eta_2\eta_2'+\eta_3\eta_3'\right)\right),
\end{aligned}
$$

where s^{*} denotes the fractional arc length parameter given by

$$
s^{*}=\left(\frac{\alpha^{2}}{\Gamma(2-\alpha)}\,u(s)\right)^{\frac{1}{\alpha}}, \tag{23}
$$

and

$$
u(s)=\int_{0}^{s}\left\|\frac{d\beta_{III}(\xi)}{d\xi}\right\|d\xi. \tag{24}
$$

Proof. Let s^{*} be the α-fractional arc length parameterization of the third-type Smarandache curve defined above. Differentiating both sides of the curve equation with respect to s yields

$$
\frac{d\beta_{III}(s^{*})}{ds^{*}}\frac{ds^{*}}{ds}=\frac{1}{\sqrt{2}}\left(\frac{d\mathcal{E}_{2}^{\alpha}(s)}{ds}+\frac{d\mathcal{E}_{3}^{\alpha}(s)}{ds}\right). \tag{25}
$$

Using the relation

$$
\frac{d\beta_{III}(s^{*})}{ds^{*}}=\frac{\Gamma(2-\alpha)}{\alpha s^{*1-\alpha}}\frac{d^{\alpha}\beta_{III}(s^{*})}{ds^{*\alpha}},
$$

Equation (25) becomes

$$
\frac{\Gamma(2-\alpha)}{\alpha s^{*1-\alpha}}\,{}_{III}\mathbb{E}_{1}^{\alpha}(s^{*})\frac{ds^{*}}{ds}=\frac{1}{\sqrt{2}}\left(-\kappa_{\alpha}(s)\mathcal{E}_{1}^{\alpha}(s)-\tau_{\alpha}(s)\mathcal{E}_{2}^{\alpha}(s)+\tau_{\alpha}(s)\mathcal{E}_{3}^{\alpha}(s)\right). \tag{26}
$$

Taking the norm of both sides of Equation (26) gives

$$
\frac{\Gamma(2-\alpha)}{\alpha s^{*1-\alpha}}\frac{ds^{*}}{ds}=\sqrt{\frac{(\kappa_{\alpha}(s))^{2}+2(\tau_{\alpha}(s))^{2}}{2}}.
$$

Substituting this result into Equation (26) yields the fractional tangent vector

$$
{}_{III}\mathbb{E}_{1}^{\alpha}(s^{*})=\frac{-\kappa_{\alpha}(s)\mathcal{E}_{1}^{\alpha}(s)-\tau_{\alpha}(s)\mathcal{E}_{2}^{\alpha}(s)+\tau_{\alpha}(s)\mathcal{E}_{3}^{\alpha}(s)}{\sqrt{(\kappa_{\alpha}(s))^{2}+2(\tau_{\alpha}(s))^{2}}}. \tag{27}
$$

Differentiating both sides of Equation (27) with respect to s leads to

$$\frac{d_{\text{III}}\mathbb{E}_1^{\alpha}(s^*)}{ds^*}\frac{ds^*}{ds} = \frac{\eta_1\mathcal{E}_1^{\alpha}(s) + \eta_2\mathcal{E}_2^{\alpha}(s) + \eta_3\mathcal{E}_3^{\alpha}(s)}{(\kappa^2(s) + 2\tau^2(s))^{3/2}},$$

where η_1, η_2, η_3 are as given in above. Furthermore, using the expression

$$\frac{ds^*}{ds} = \frac{\sqrt{\kappa^2(s) + 2\tau^2(s)}}{(s^*/s)^{\alpha-1}\sqrt{2}},$$

we obtain

$$\frac{d_{\text{III}}\mathbb{E}_1^{\alpha}(s^*)}{ds^*} = \frac{(s^*/s)^{\alpha-1}\sqrt{2}}{\left(\kappa^2(s) + 2\tau^2(s)\right)^2}\left(\eta_1\mathcal{E}_1^{\alpha}(s) + \eta_2\mathcal{E}_2^{\alpha}(s) + \eta_3\mathcal{E}_3^{\alpha}(s)\right).$$

Consequently, the curvature function of the first-type fractional Smarandache curve is calculated as

$${}_{III}\kappa_{\alpha}^*(s) = \left\|\frac{d_{\text{III}}\mathbb{E}_1^{\alpha}(s^*)}{ds^*}\right\| = \frac{(s^*/s)^{\alpha-1}\sqrt{2}}{\left(\kappa^2(s) + 2\tau^2(s)\right)^2}\sqrt{(\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2},$$

where s^* is given by Equation (23). The fractional normal vector is then found as

$${}_{\text{III}}\mathbb{E}_2^{\alpha}(s^*) = \frac{d_{\text{III}}\mathbb{E}_1^{\alpha}(s^*)/ds^*}{\left\|d_{\text{III}}\mathbb{E}_1^{\alpha}(s^*)/ds^*\right\|} = \frac{\eta_1\mathcal{E}_1^{\alpha}(s) + \eta_2\mathcal{E}_2^{\alpha}(s) + \eta_3\mathcal{E}_3^{\alpha}(s)}{\sqrt{(\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2}}.$$

The fractional binormal vector is obtained via the cross product

$$\begin{aligned}{}_{\text{III}}\mathbb{E}_3^{\alpha}(s^*) = {}_{\text{III}}\mathbb{E}_1^{\alpha}(s^*) \wedge {}_{\text{III}}\mathbb{E}_2^{\alpha}(s^*) \\ (-\tau_{\alpha}(s)\eta_3 - \tau_{\alpha}(s)\eta_2)\mathcal{E}_1^{\alpha}(s) \\ + (\kappa_{\alpha}(s)\eta_3 + \tau_{\alpha}(s)\eta_1)\mathcal{E}_2^{\alpha}(s) \\ + (-\kappa_{\alpha}(s)\eta_2 + \tau_{\alpha}(s)\eta_1)\mathcal{E}_3^{\alpha}(s) \\ = \frac{}{\sqrt{\left((\kappa_{\alpha}(s))^2 + 2(\tau_{\alpha}(s))^2\right)\left((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2\right)}}.\end{aligned}$$

Finally, the fractional torsion function is given by

$$\begin{aligned}{}_{III}\tau_{\alpha}^*(s) &= \left\langle \frac{d_{\text{III}}\mathbb{E}_2^{\alpha}(s^*)}{ds^*}, {}_{\text{III}}\mathbb{E}_3^{\alpha}(s^*)\right\rangle \\ &= \frac{(s^*/s)^{\alpha-1}\sqrt{2}}{\left(\kappa^2(s) + 2\tau^2(s)\right)\left((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2\right)^2}\left(B_1 + B_2 + B_3\right)\end{aligned}$$

where B_1, B_2, B_3 are as defined above. □

Definition 8. *Let β be a unit-speed fractional curve parameterized by α-arc length in three-dimensional Euclidean space, and let $\{\mathcal{E}_1^{\alpha}(s), \mathcal{E}_2^{\alpha}(s), \mathcal{E}_3^{\alpha}(s)\}$ be its fractional Frenet frame apparatus. The $\mathcal{E}_1^{\alpha}(s)\mathcal{E}_2^{\alpha}(s)\mathcal{E}_3^{\alpha}(s)$-fractional Smarandache curve is defined by*

$$\beta_{IV}(s) = \frac{\mathcal{E}_1^{\alpha}(s) + \mathcal{E}_2^{\alpha}(s) + \mathcal{E}_3^{\alpha}(s)}{\sqrt{3}}, \tag{28}$$

where $0 < \alpha \leq 1$ and $s > 0$.

Theorem 4. *Let $\beta(s) : I \to \mathbb{E}^3$ be a unit-speed fractional curve parameterized by α-arc length in three-dimensional Euclidean space. The Frenet frame of the fourth-type fractional Smarandache curve is given by*

$$
\begin{aligned}
{}_{IV}\mathbb{E}_1^{\alpha}(s) &= \frac{-\kappa(s)\mathcal{E}_1^{\alpha}(s) + (\kappa(s)-\tau(s))\mathcal{E}_2^{\alpha}(s) + \tau(s)\mathcal{E}_3^{\alpha}(s)}{\sqrt{2\big(\kappa^2(s)+\tau^2(s)-\kappa(s)\tau(s)\big)}},\\
{}_{IV}\mathbb{E}_2^{\alpha}(s) &= \frac{\mu_1\mathcal{E}_1^{\alpha}(s) + \mu_2\mathcal{E}_2^{\alpha}(s) + \mu_3\mathcal{E}_3^{\alpha}(s)}{\sqrt{(\mu_1)^2+(\mu_2)^2+(\mu_3)^2}},\\
{}_{IV}\mathbb{E}_3^{\alpha}(s) &= \frac{\begin{array}{l}\big((\kappa(s)-\tau(s))\mu_3 - \tau(s)\mu_2\big)\mathcal{E}_1^{\alpha}(s)\\ +\big(\kappa(s)\mu_3+\tau(s)\mu_1\big)\mathcal{E}_2^{\alpha}(s)\\ +\big(-\kappa(s)\mu_2-(\kappa(s)-\tau(s))\mu_1\big)\mathcal{E}_3^{\alpha}(s)\end{array}}{\sqrt{2\big(\kappa^2(s)+\tau^2(s)-\kappa(s)\tau(s)\big)\big((\mu_1)^2+(\mu_2)^2+(\mu_3)^2\big)}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_1 &= \kappa\big(2\kappa\kappa' + 2\tau\tau' - \kappa'\tau - \kappa\tau'\big) + 2\big(\kappa^2+\tau^2-\kappa\tau\big)\big(-\kappa' - (\kappa-\tau)\kappa_{\alpha}\big),\\
\mu_2 &= (\tau-\kappa)\big(2\kappa\kappa' + 2\tau\tau' - \kappa'\tau - \kappa\tau'\big) + 2\big(\kappa^2+\tau^2-\kappa\tau\big)\big(-\kappa\kappa_{\alpha} - \tau\tau_{\alpha} + \kappa' - \tau'\big),\\
\mu_3 &= -\tau\big(2\kappa\kappa' + 2\tau\tau' - \kappa'\tau - \kappa\tau'\big) + 2\big(\kappa^2+\tau^2-\kappa\tau\big)\big(\tau' + (\kappa-\tau)\tau_{\alpha}\big),
\end{aligned}
$$

and the prime denotes differentiation with respect to the integer-order parameter s. Furthermore, its curvature and torsion functions are obtained as

$$
\begin{aligned}
{}_{IV}\kappa_{\alpha}^{*}(s) &= \frac{\sqrt{3}\big(\frac{s^*}{s}\big)^{\alpha-1}}{4\big(\kappa^2(s)+\tau^2(s)-\kappa(s)\tau(s)\big)^2}\sqrt{(\mu_1)^2+(\mu_2)^2+(\mu_3)^2},\\
{}_{IV}\tau_{\alpha}^{*}(s) &= \frac{\sqrt{3}\big(\frac{s^*}{s}\big)^{\alpha-1}}{2\big(\kappa^2(s)+\tau^2(s)-\kappa(s)\tau(s)\big)\big((\mu_1)^2+(\mu_2)^2+(\mu_3)^2\big)^2}\big(C_1+C_2+C_3\big),
\end{aligned}
$$

with

$$
\begin{aligned}
C_1 &= \Big(\big(\kappa(s)-\tau(s)\big)\mu_3 - \tau(s)\mu_2\Big)\\
&\quad\times\Big(\big((\mu_1)^2+(\mu_2)^2+(\mu_3)^2\big)\big(\mu_1' - \mu_2\kappa_{\alpha}(s)\big) + \mu_1\big(\mu_1\mu_1' + \mu_2\mu_2' + \mu_3\mu_3'\big)\Big),\\
C_2 &= \Big(\kappa(s)\mu_3 + \tau(s)\mu_1\Big)\\
&\quad\times\Big(\big((\mu_1)^2+(\mu_2)^2+(\mu_3)^2\big)\big(\mu_1\kappa_{\alpha}(s) + \mu_1' - \mu_3\tau_{\alpha}(s)\big) + \mu_2\big(\mu_1\mu_1' + \mu_2\mu_2' + \mu_3\mu_3'\big)\Big),\\
C_3 &= \Big(\big(\tau(s)-\kappa(s)\big)\mu_1 - \kappa(s)\mu_2\Big)\\
&\quad\times\Big(\big((\mu_1)^2+(\mu_2)^2+(\mu_3)^2\big)\big(\mu_3' + \mu_2\tau_{\alpha}(s)\big) + \mu_3\big(\mu_1\mu_1' + \mu_2\mu_2' + \mu_3\mu_3'\big)\Big),
\end{aligned}
$$

where s^* denotes the fractional arc length parameter given by

$$
s^* = \left(\frac{\alpha^2}{\Gamma(2-\alpha)}\,u(s)\right)^{\frac{1}{\alpha}}, \tag{29}
$$

and

$$
u(s) = \int_0^s \left\| \frac{d\beta_{IV}(\xi)}{d\xi} \right\| d\xi. \tag{30}
$$

Proof. Let s^* be the α-fractional arc length parameterization of the fourth-type Smarandache curve defined above. Differentiating both sides of the curve equation with respect to s yields

$$\frac{d\beta_{IV}(s^*)}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(\frac{d\mathcal{E}_1^\alpha(s)}{ds} + \frac{d\mathcal{E}_2^\alpha(s)}{ds} + \frac{d\mathcal{E}_3^\alpha(s)}{ds}\right). \tag{31}$$

Using the relation

$$\frac{d\beta_{IV}(s^*)}{ds^*} = \frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}\frac{d^\alpha \beta_{IV}(s^*)}{d{s^*}^\alpha},$$

Equation (31) becomes

$$\frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}{}_{\mathbb{IV}}\mathbb{E}_1^\alpha(s^*)\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-\kappa_\alpha(s)\mathcal{E}_1^\alpha(s) + (\kappa_\alpha(s)-\tau_\alpha(s))\mathcal{E}_2^\alpha(s) + \tau_\alpha(s)\mathcal{E}_3^\alpha(s)). \tag{32}$$

Taking the norm of both sides of Equation (32) gives

$$\frac{\Gamma(2-\alpha)}{\alpha {s^*}^{1-\alpha}}\frac{ds^*}{ds} = \sqrt{\frac{2(\kappa_\alpha(s))^2 + 2(\tau_\alpha(s))^2 - 2\kappa_\alpha(s)\tau_\alpha(s)}{3}}.$$

Substituting this result into Equation (32) yields the fractional tangent vector

$${}_{\mathbb{IV}}\mathbb{E}_1^\alpha(s^*) = \frac{-\kappa_\alpha(s)\mathcal{E}_1^\alpha(s) + (\kappa_\alpha(s)-\tau_\alpha(s))\mathcal{E}_2^\alpha(s) + \tau_\alpha(s)\mathcal{E}_3^\alpha(s)}{\sqrt{2(\kappa_\alpha(s))^2 + 2(\tau_\alpha(s))^2 - 2\kappa_\alpha(s)\tau_\alpha(s)}}. \tag{33}$$

Differentiating both sides of Equation (33) with respect to s leads to

$$\frac{d_{\mathbb{IV}}\mathbb{E}_1^\alpha(s^*)}{ds^*}\frac{ds^*}{ds} = \frac{\mu_1\mathcal{E}_1^\alpha(s) + \mu_2\mathcal{E}_2^\alpha(s) + \mu_3\mathcal{E}_3^\alpha(s)}{(2\kappa^2(s) + 2\tau^2(s) - 2\kappa(s)\tau(s))^{3/2}},$$

where η_1, η_2, η_3 are as given in above. Furthermore, using the expression

$$\frac{ds^*}{ds} = \frac{\sqrt{2(\kappa(s))^2 + 2(\tau(s))^2 - 2\kappa(s)\tau(s)}}{(s^*/s)^{\alpha-1}\sqrt{3}},$$

we obtain

$$\frac{d_{\mathbb{IV}}\mathbb{E}_1^\alpha(s^*)}{ds^*} = \frac{(s^*/s)^{\alpha-1}\sqrt{3}}{\left(2\kappa^2(s) + 2\tau^2(s) - 2\kappa(s)\tau(s)\right)^2}\left(\mu_1\mathcal{E}_1^\alpha(s) + \mu_2\mathcal{E}_2^\alpha(s) + \mu_3\mathcal{E}_3^\alpha(s)\right).$$

Consequently, the curvature function of the first-type fractional Smarandache curve is calculated as

$${}_{IV}\kappa_\alpha^*(s) = \left\|\frac{d_{\mathbb{IV}}\mathbb{E}_1^\alpha(s^*)}{ds^*}\right\| = \frac{(s^*/s)^{\alpha-1}\sqrt{3}}{\left(2\kappa^2(s) + 2\tau^2(s) - 2\kappa(s)\tau(s)\right)^2}\sqrt{(\mu_1)^2 + (\mu_2)^2 + (\mu_3)^2},$$

where s^* is given by Equation (29). The fractional normal vector is then found as

$${}_{\mathbb{IV}}\mathbb{E}_2^\alpha(s^*) = \frac{d_{\mathbb{IV}}\mathbb{E}_1^\alpha(s^*)/ds^*}{\left\|d_{\mathbb{IV}}\mathbb{E}_1^\alpha(s^*)/ds^*\right\|} = \frac{\mu_1\mathcal{E}_1^\alpha(s) + \mu_2\mathcal{E}_2^\alpha(s) + \mu_3\mathcal{E}_3^\alpha(s)}{\sqrt{(\mu_1)^2 + (\mu_2)^2 + (\mu_3)^2}}.$$

The fractional binormal vector is obtained via the cross product

$$\begin{aligned}{}_{\mathbb{IV}}\mathbb{E}_3^{\alpha}(s^*) &= {}_{\mathbb{IV}}\mathbb{E}_1^{\alpha}(s^*) \wedge {}_{\mathbb{IV}}\mathbb{E}_2^{\alpha}(s^*) \\ &\quad ((\kappa(s)-\tau(s))\mu_3 - \tau(s)\mu_2)\mathcal{E}_1^{\alpha}(s) \\ &\quad + (\kappa(s)\mu_3 + \tau(s)\mu_1)\mathcal{E}_2^{\alpha}(s) \\ &\quad + (-\kappa(s)\mu_2 - (\kappa(s)-\tau(s))\mu_1)\mathcal{E}_3^{\alpha}(s) \\ &= \frac{}{\sqrt{\left((2\kappa_(s))^2 + 2(\tau(s))^2 - 2\kappa(s)\tau(s)\right)\left((\mu_1)^2 + (\mu_2)^2 + (\mu_3)^2\right)}}.\end{aligned}$$

Finally, the fractional torsion function is given by

$$\begin{aligned}{}_{IV}\tau_{\alpha}^{*}(s) &= \left\langle \frac{d_{\mathbb{IV}}\mathbb{E}_2^{\alpha}(s^*)}{ds^*}, {}_{\mathbb{IV}}\mathbb{E}_3^{\alpha}(s^*) \right\rangle \\ &= \frac{(s^*/s)^{\alpha-1}\sqrt{3}}{\left(2\kappa^2(s) + 2\tau^2(s) - 2\kappa^2(s)\tau^2(s)\right)\left((\mu_1)^2 + (\mu_2)^2 + (\mu_3)^2\right)^2}(C_1 + C_2 + C_3)\end{aligned}$$

where C_1, C_2, C_3 are as defined above. □

Example 1. *Let $\beta : I \subset \mathbb{R}^+ \to \mathbb{E}^3$ be a unit-speed fractional curve in $\mathbb{R}^3$ parametrized by*

$$\begin{aligned}\beta(s) = \Bigg(&4\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) + 4\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right), \\ &4\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - 4\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right), \\ &\frac{\Gamma(2-\alpha)s^{\alpha}}{3\alpha^2}\Bigg),\end{aligned}$$

where $0 < \alpha \leq 1$. Taking the Caputo fractional derivative of order α of the curve $\beta(s)$ yields

$$\begin{aligned}\frac{d^{\alpha}\beta(s)}{ds^{\alpha}} &= \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\frac{d\beta(s)}{ds} \\ &= \Bigg(\frac{2\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - 2\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right)}{3}, \\ &\quad \frac{2\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) + 2\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right)}{3}, \frac{1}{3}\Bigg).\end{aligned}$$

Thus, the curve is unit-speed and parametrized with respect to the fractional order α; accordingly, the fractional tangent, principal normal, and binormal vectors are obtained as

$$\begin{aligned}
\mathcal{E}_1^{\alpha}(s) &= \frac{d^{\alpha}\beta(s)}{ds^{\alpha}} = \frac{1}{3}\left(2\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - 2\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\right.\\
&\qquad \left. 2\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) + 2\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\ 1\right),\\
\mathcal{E}_2^{\alpha}(s) &= \frac{\frac{d}{ds}(\mathcal{E}_1^{\alpha}(s))}{\left\|\frac{d}{ds}(\mathcal{E}_1^{\alpha}(s))\right\|} = \frac{1}{\sqrt{2}}\left(-\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\right.\\
&\qquad \left. -\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) + \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\ 0\right),\\
\mathcal{E}_3^{\alpha}(s) &= \mathcal{E}_1^{\alpha}(s) \wedge \mathcal{E}_2^{\alpha}(s) = \frac{1}{3\sqrt{2}}\left(\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\right.\\
&\qquad \left. -\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\ 0\right).
\end{aligned} \tag{34}$$

Moreover, the fractional curvature and torsion functions are found to be

$$\kappa_{\alpha}(s) = \left\|\frac{d(\mathcal{E}_1^{\alpha}(s))}{ds}\right\| = \frac{\sqrt{2}}{9}\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}, \qquad \tau_{\alpha}(s) = \left\langle\frac{d\mathcal{E}_2^{\alpha}(s)}{ds}, \mathcal{E}_3^{\alpha}(s)\right\rangle = \frac{1}{18}\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}. \tag{35}$$

The fractional Frenet frames and curvature functions of the first, second, third, and fourth-type Smarandache curves of fractional order are calculated as follows.

First-Type: According to Equation (10), the first-type fractional Smarandache curve is defined as

$$\begin{aligned}
\beta_I(s) &= \frac{1}{6}\left((2\sqrt{2}-3)\cos(\varphi) - (2\sqrt{2}+3)\sin(\varphi),\right.\\
&\qquad \left.(2\sqrt{2}-3)\sin(\varphi) + (2\sqrt{2}+3)\cos(\varphi),\ \sqrt{2}\right),
\end{aligned} \tag{36}$$

where $\varphi = \frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}$. *Taking the integer-order derivative of Equation (36) with respect to s and then computing its norm gives*

$$\left\|\frac{d\beta_I(s)}{ds}\right\| = \frac{\sqrt{34}\,\Gamma(2-\alpha)s^{\alpha-1}}{36\alpha}.$$

Consequently, the equalities given in (12) and (13) yield

$$\begin{aligned}
u(s) &= \int_0^s \frac{\sqrt{34}\,\Gamma(2-\alpha)\xi^{\alpha-1}}{36\alpha}\,d\xi = \frac{\sqrt{34}\,\Gamma(2-\alpha)s^{\alpha}}{36\alpha^2},\\
s^* &= \left(\frac{\sqrt{34}}{36}s^{\alpha}\right)^{\frac{1}{\alpha}}.
\end{aligned} \tag{37}$$

Thus, the fractional curve given in (36) can be rewritten with respect to the fractional arc-length parameterization s^* as

$$\beta_I(s^*) = \frac{1}{6}\Bigg((2\sqrt{2}-3)\cos(\theta) - (2\sqrt{2}+3)\sin(\theta), \\ (2\sqrt{2}-3)\sin(\theta) + (2\sqrt{2}+3)\cos(\theta),\ \sqrt{2}\Bigg), \tag{38}$$

where $\theta = \frac{6\Gamma(2-\alpha)(s^*)^\alpha}{\sqrt{34}\,\alpha^2}$. It is straightforward to verify that $\left\|\frac{d^\alpha \beta(s^*)}{ds^{*\alpha}}\right\| = 1$. Hence, using the relation between s and s^* given in (37), the fractional tangent, principal normal, and binormal vectors of the first-type fractional Smarandache curve are found as

$$\begin{aligned}
{}_{\mathbb{I}}\mathbb{E}_1^\alpha(s) &= \frac{1}{\sqrt{34}}\Bigg(-(2\sqrt{2}-3)\sin(\varphi) - (2\sqrt{2}+3)\cos(\varphi), \\ &\qquad (2\sqrt{2}-3)\cos(\varphi) - (2\sqrt{2}+3)\sin(\varphi),\ 0\Bigg), \\
{}_{\mathbb{I}}\mathbb{E}_2^\alpha(s) &= \frac{1}{\sqrt{34}}\Bigg(-(2\sqrt{2}-3)\cos(\varphi) + (2\sqrt{2}+3)\sin(\varphi), \\ &\qquad -(2\sqrt{2}-3)\sin(\varphi) - (2\sqrt{2}+3)\cos(\varphi),\ 0\Bigg), \\
{}_{\mathbb{I}}\mathbb{E}_3^\alpha(s) &= (0,0,1),
\end{aligned} \tag{39}$$

where $\varphi = \frac{\Gamma(2-\alpha)s^\alpha}{6\alpha^2}$. Moreover, the fractional curvature and torsion functions are obtained as

$$\begin{aligned}
{}_I\kappa_\alpha^*(s^*) &= \frac{6}{\sqrt{34}}\frac{\Gamma(2-\alpha)}{\alpha s^{*1-\alpha}} \quad \Rightarrow \quad {}_I\kappa_\alpha^*(s) = \left(\frac{36}{\sqrt{34}}\right)^{\frac{1}{\alpha}}\frac{\Gamma(2-\alpha)}{6\alpha s^{1-\alpha}}, \\
{}_I\tau_\alpha^*(s) &= 0.
\end{aligned} \tag{40}$$

Second-Type: According to Equation (17), the second-type fractional Smarandache curve is defined as

$$\beta_{II}(s) = \frac{2\sqrt{2}-1}{6}\Bigg(\cos\left(\frac{\Gamma(2-\alpha)s^\alpha}{6\alpha^2}\right) - \sin\left(\frac{\Gamma(2-\alpha)s^\alpha}{6\alpha^2}\right), \\ \cos\left(\frac{\Gamma(2-\alpha)s^\alpha}{6\alpha^2}\right) + \sin\left(\frac{\Gamma(2-\alpha)s^\alpha}{6\alpha^2}\right), \\ \frac{8+9\sqrt{2}}{7}\Bigg).$$

The fractional tangent, principal normal, and binormal vectors of the second-type fractional Smarandache curve are then given by

$$
\begin{aligned}
{}_{II}\mathbb{E}_1^{\alpha}(s) &= \frac{1}{\sqrt{2}}\Bigg(-\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right), \\
&\qquad \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - \sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\ 0\Bigg), \\
{}_{II}\mathbb{E}_2^{\alpha}(s) &= \frac{1}{\sqrt{2}}\Bigg(-\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) + \sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right), \\
&\qquad -\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\ 0\Bigg), \\
{}_{II}\mathbb{E}_3^{\alpha}(s) &= (0,0,1).
\end{aligned}
$$

Furthermore, the fractional curvature and torsion functions are obtained as

$$
\begin{aligned}
{}_{II}\kappa_{\alpha}^{*}(s^{*}) &= \frac{6}{4-\sqrt{2}}\frac{\Gamma(2-\alpha)}{\alpha s^{*1-\alpha}} \quad \Rightarrow \quad {}_{II}\kappa_{\alpha}^{*}(s) = \left(\frac{36}{4-\sqrt{2}}\right)^{\frac{1}{\alpha}}\frac{\Gamma(2-\alpha)}{6\alpha s^{1-\alpha}}, \\
{}_{II}\tau_{\alpha}^{*}(s) &= 0.
\end{aligned}
$$

Third-Type: According to Equation (22), the third-type fractional Smarandache curve is defined as

$$
\begin{aligned}
\beta_{III}(s) = \frac{1}{3}\Bigg(&-\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - 2\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right), \\
&-2\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) + \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\ 2\Bigg).
\end{aligned}
$$

The fractional tangent, principal normal, and binormal vectors of the third-type fractional Smarandache curve are calculated as

$$
\begin{aligned}
{}_{III}\mathbb{E}_1^{\alpha}(s) &= \frac{1}{\sqrt{5}}\Bigg(-\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) + 2\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right), \\
&\qquad -2\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - \sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\ 0\Bigg), \\
{}_{III}\mathbb{E}_2^{\alpha}(s) &= \frac{1}{\sqrt{5}}\Bigg(\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) + 2\cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right), \\
&\qquad 2\sin\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right) - \cos\left(\frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}\right),\ 0\Bigg), \\
{}_{III}\mathbb{E}_3^{\alpha}(s) &= (0,0,1).
\end{aligned}
$$

In addition, the fractional curvature and torsion functions are found to be

$$
\begin{aligned}
{}_{III}\kappa_{\alpha}^{*}(s^{*}) &= \frac{3}{\sqrt{5}}\frac{\Gamma(2-\alpha)}{\alpha s^{*1-\alpha}} \quad \Rightarrow \quad {}_{III}\kappa_{\alpha}^{*}(s) = \left(\frac{18}{\sqrt{5}}\right)^{\frac{1}{\alpha}}\frac{\Gamma(2-\alpha)}{6\alpha s^{1-\alpha}}, \\
{}_{III}\tau_{\alpha}^{*}(s) &= 0.
\end{aligned}
$$

Fourth-Type: According to Equation (28), the fourth-type fractional Smarandache curve is defined as

$$\beta_{IV}(s) = \frac{1}{3\sqrt{6}}\Big((2\sqrt{2}-4)\cos(\varphi) - (2\sqrt{2}+2)\sin(\varphi), \\ (2\sqrt{2}+2)\cos(\varphi) + (2\sqrt{2}-4)\sin(\varphi), \\ 4+\sqrt{2}\Big),$$

where $\varphi = \frac{\Gamma(2-\alpha)s^{\alpha}}{6\alpha^2}$. *The fractional tangent, principal normal, and binormal vectors of the fourth-type fractional Smarandache curve are calculated as*

$${}_{\mathbb{IV}}\mathbb{E}_1^{\alpha}(s) = \frac{1}{\sqrt{36-8\sqrt{2}}}\Big(-(2\sqrt{2}-4)\sin(\varphi) - (2\sqrt{2}+2)\cos(\varphi), \\ -(2\sqrt{2}+2)\sin(\varphi) + (2\sqrt{2}-4)\cos(\varphi),\ 0\Big),$$

$${}_{\mathbb{IV}}\mathbb{E}_2^{\alpha}(s) = \frac{1}{\sqrt{36-8\sqrt{2}}}\Big(-(2\sqrt{2}-4)\cos(\varphi) + (2\sqrt{2}+2)\sin(\varphi), \\ -(2\sqrt{2}+2)\cos(\varphi) - (2\sqrt{2}-4)\sin(\varphi),\ 0\Big),$$

$${}_{\mathbb{IV}}\mathbb{E}_3^{\alpha}(s) = (0,0,1).$$

Thus, the fractional curvature and torsion functions are obtained as

$${}_{IV}\kappa_{\alpha}^{*}(s^{*}) = \frac{3\sqrt{6}\,\Gamma(2-\alpha)}{\sqrt{36-8\sqrt{2}}\,\alpha s^{*1-\alpha}} \quad \Rightarrow \quad {}_{IV}\kappa_{\alpha}^{*}(s) = \left(\frac{18\sqrt{3}}{\sqrt{18-4\sqrt{2}}}\right)^{\frac{1}{\alpha}}\frac{\Gamma(2-\alpha)}{6\alpha s^{1-\alpha}},$$
$${}_{IV}\tau_{\alpha}^{*}(s) = 0.$$

Remark 1. *Consistency with the classical special case* $\alpha = 1$*: For the curve* $\beta(s)$*, which is parametrized by the fractional arc-length parameterization given in the example 1, taking* $\alpha = 1$ *reduces the curve to the following form:*

$$\beta(s) = \left(4\sin\left(\frac{s}{6}\right) + 4\cos\left(\frac{s}{6}\right),\ 4\sin\left(\frac{s}{6}\right) - 4\cos\left(\frac{s}{6}\right),\ \frac{s}{3}\right).$$

Accordingly, the tangent, principal normal and binormal vectors of this curve are obtained, respectively, as

$$T(s) = \frac{1}{3}\left(2\cos\left(\frac{s}{6}\right) - 2\sin\left(\frac{s}{6}\right),\ 2\cos\left(\frac{s}{6}\right) + 2\sin\left(\frac{s}{6}\right),\ 1\right),$$
$$N(s) = \frac{1}{\sqrt{2}}\left(-\sin\left(\frac{s}{6}\right) - \cos\left(\frac{s}{6}\right),\ -\sin\left(\frac{s}{6}\right) + \cos\left(\frac{s}{6}\right),\ 0\right), \tag{41}$$
$$B(s) = \frac{1}{3\sqrt{2}}\left(\sin\left(\frac{s}{6}\right) - \cos\left(\frac{s}{6}\right),\ -\sin\left(\frac{s}{6}\right) - \cos\left(\frac{s}{6}\right),\ 0\right).$$

Thus, the curvature and torsion functions are given by

$$\kappa(s) = \frac{\sqrt{2}}{9}, \qquad \tau(s) = \frac{1}{18}. \tag{42}$$

respectively. Consequently, the standard first-type Smarandache curve is defined by

$$\beta_{I}(s) = \frac{1}{6}\Big((2\sqrt{2}-3)\cos\Big(\frac{s}{6}\Big) - (2\sqrt{2}+3)\sin\Big(\frac{s}{6}\Big), (2\sqrt{2}-3)\sin\Big(\frac{s}{6}\Big) + (2\sqrt{2}+3)\cos\Big(\frac{s}{6}\Big), \sqrt{2}\Big),$$

Also, the tangent, principal normal, and binormal vectors of the first-type fractional Smarandache curve are calculated as

$$\begin{aligned}
{}_{I}\mathbb{E}_{1}(s) &= \frac{1}{\sqrt{34}}\Bigg(-(2\sqrt{2}-3)\sin\Big(\frac{s}{6}\Big) - (2\sqrt{2}+3)\cos\Big(\frac{s}{6}\Big), \\
&\qquad (2\sqrt{2}-3)\cos\Big(\frac{s}{6}\Big) - (2\sqrt{2}+3)\sin\Big(\frac{s}{6}\Big),\ 0\Bigg), \\
{}_{I}\mathbb{E}_{2}(s) &= \frac{1}{\sqrt{34}}\Bigg(-(2\sqrt{2}-3)\cos\Big(\frac{s}{6}\Big) + (2\sqrt{2}+3)\sin\Big(\frac{s}{6}\Big), \\
&\qquad -(2\sqrt{2}-3)\sin\Big(\frac{s}{6}\Big) - (2\sqrt{2}+3)\cos\Big(\frac{s}{6}\Big),\ 0\Bigg), \\
{}_{I}\mathbb{E}_{3}(s) &= (0,0,1).
\end{aligned} \tag{43}$$

Moreover, the fractional curvature and torsion functions are obtained as

$${}_{I}\kappa(s) = \frac{6}{\sqrt{34}},\ {}_{I}\tau(s) = 0.$$

In addition, the reason why the torsion vanishes for all four types of Smarandache curves is that the selected curve is a helix. Also, As can be seen, when $\alpha = 1$ is taken in Equation (34), the corresponding expression reduces to Equation (41). Similarly, setting $\alpha = 1$ in Equation (35) yields the classical form of the curvature functions given in Equation (42). Moreover, when $\alpha = 1$ is substituted into Equation (39), which represents the fractional Frenet vectors of the first-type fractional Smarandache curve, these vectors coincide with the standard Frenet frame given in Equation (43). The same compatibility also holds for the curvature functions. Therefore, the special case where $\alpha = 1$ clearly demonstrates that the resulting fractional structure provides a consistent generalization that encompasses the classical theory. Similarly, it can be easily shown that the same correspondence holds for the second, third, and fourth types of Smarandache curves. Moreover, as can be seen from the figure below, it is clearly evident that as the value of α approaches 1, the fractional curvature function numerically converges to the classical curvature function. Thus, in addition to the algebraic consistency, it is demonstrated that the transition between the fractional structure and the classical structure is also numerically consistent. In Figure 1, graphical representations of the curvature functions of the first-type, second-type, third-type, and fourth-type fractional Smarandache curves are presented for the values $\alpha = 1$ (black), $\alpha = 0.99$ (red), $\alpha = 0.88$ (blue), and $\alpha = 0.77$ (green).

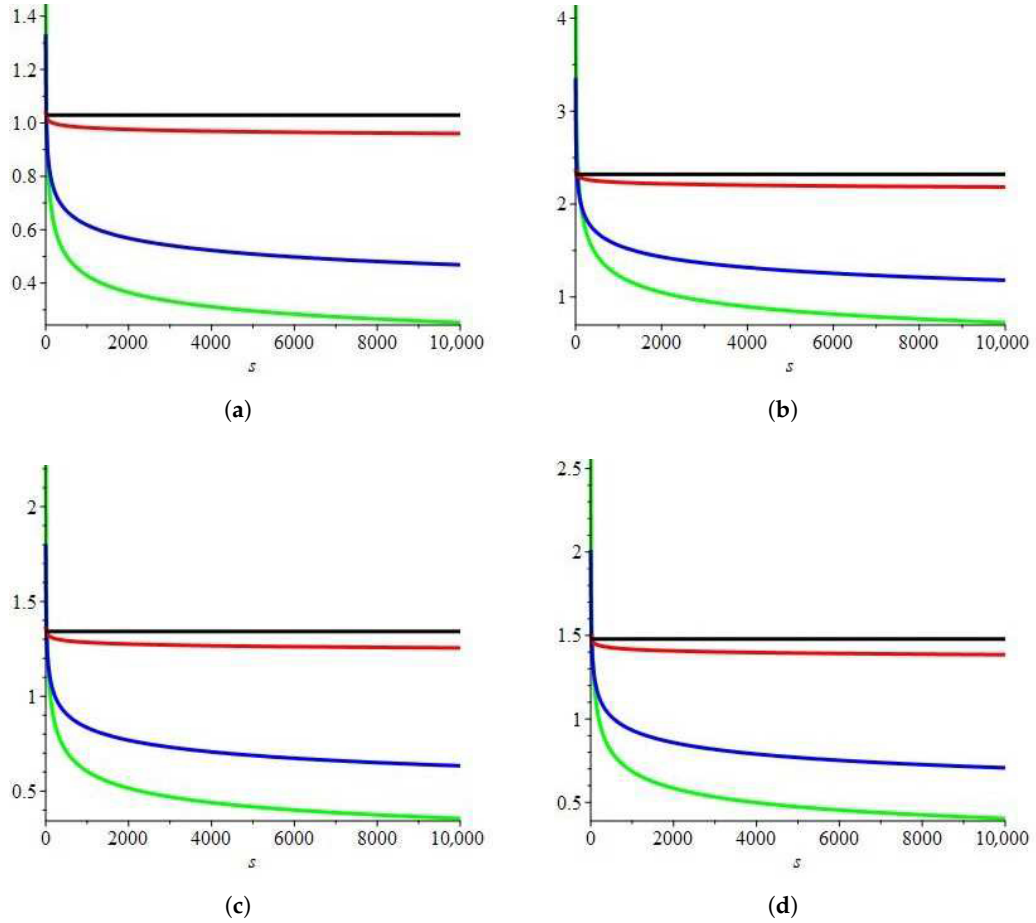

Figure 1. Fractional curvature function of the Smarandache curve for different values of α. **(a)** ${}_{I}\kappa_{\alpha}^{*}(s)$; **(b)** ${}_{II}\kappa_{\alpha}^{*}(s)$; **(c)** ${}_{III}\kappa_{\alpha}^{*}(s)$; **(d)** ${}_{IV}\kappa_{\alpha}^{*}(s)$.

4. Conclusions

In classical differential geometry, the intrinsic properties of curves are studied through their Frenet frames and associated curvature functions. This study investigates in detail the characterization of the Smarandache curve defined under fractional arc-length parameterization. In this context, the fractional tangent, principal normal, and binormal vectors, together with the fractional curvature functions of the curve considered in the example, are obtained, and the the first, second, third, and fourth types of fractional Smarandache curves are defined accordingly. Moreover the fractional Frenet vectors and curvature functions associated with these curves are derived explicitly.

As a result, it is observed that the fractional curvature functions exhibit a deviation from the classical case depending on the parameter α, whereas, in the special case $\alpha = 1$ all structures coincide consistently with the classical theory. In this context, the reduction of the fractional Frenet vectors to the classical Frenet frame, and of the fractional Smarandache curves to the standard Smarandache curves, demonstrates that the obtained result constitute a consistent generalization encompassing the classical theory. Moreover, it is seen in Figure 1 that this algebraically established correspondence is also supported at the numerical level. In particular, as the value of α approaches 1, the fractional curvature functions are observed to converge numerically to the classical curvature functions. This shows that the transition between the fractional classical structures is coherent not only from a theoretical perspective but also at the numerically.

That the fractional-order derivative operator be reformulated in a way that preserves certain classical differential properties of fundamental importance in curve and surface theory,

particularly the chain rule, the Leibniz rule, and the vanishing derivative of constants. Such an approach may allow the influence of fractional derivatives on geometric structures to be examined within a more natural and comprehensive. In this way, fractional differential geometry may be placed on a stronger theoretical foundation with respect to both curve theory and surface theory.

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